The Zeta Function

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Definition and history

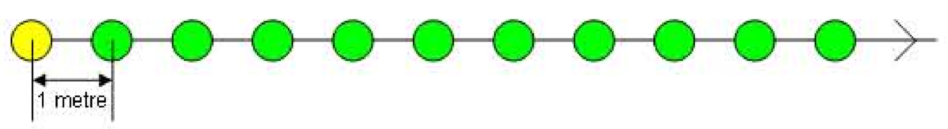
An infinite series is a sum containing infinitely many terms. The Euler zeta function, or simply the zeta function (as we shall call it), is an infinite series and is defined as follows:

$$\zeta(s)=1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

where s is any real number greater than 1 (a real number is an integer, a rational number or an irrational number – we shall see why we require s to be greater than 1 later). The number n in the sum runs through all the positive integers (whole numbers e.g. 1, 2, 3) and ζ is the sixth letter of the Greek alphabet which is named zeta.
The zeta function was first formally used by Leonhard Euler (1707-83) in 1737. Euler regarded s as a positive integer greater than 1; since then the definition of the zeta function has been extended to the one given above. The zeta function has various remarkable properties, some of which we shall explore. It also lies behind one of the current great unsolved problem in mathematics – the Riemann hypothesis.

Physical problem

Suppose we have infinitely many particles lying on the same straight line. Each particle has mass 1 kg and the distance between each pair of successive particles is 1 metre (this is illustrated below).



We wish to obtain the magnitude of the force (measured in Newtons) exerted on the yellow particle due to the infinitely many green particles along the straight line (each green particle will exert an attractive force on the yellow particle as all masses exert an attractive force on all other masses).

Questions we might ask about this force are:
- Will the force on the yellow particle grow without bound as we sum the attractive force exerted on it by each successive green particle?
- Alternatively, will the force exerted on the yellow particle approach some finite value as we sum the force exerted on it by each successive green particle, and if so, what is this value?

Sir Isaac Newton's universal law of gravitation states that the attractive force two bodies exert on each other is equal to $\frac{Gm_1m_2}{r^2}$ where G is the universal gravitational constant which has magnitude 6.67×10^{-11}; m_1 and m_2 are the masses of the bodies and r is the distance between the two masses.
Thus the force exerted on the yellow particle by the first green particle is $\frac{G}{1^2}$ and by the second green particle is $\frac{G}{2^2}$ and so on. Hence the overall force exerted by the green particles on the yellow particle is

$$\frac{G}{1^2}+\frac{G}{2^2}+\frac{G}{3^2}+\cdots=G\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)=G\zeta(2)$$

Thus the key to answering our problem is to know and understand some of the properties of $\zeta(2)$, which we shall explain as we proceed.

Convergence

The mathematical topic of convergence (e.g. of sequences and series) is part of a branch of mathematics known as analysis. You are likely to study the topic if you do a degree in mathematics.

Definition of convergence of an infinite series:

- Take the infinite series $S_\infty=\sum_{n=1}^{\infty}a_n$ and the sum of the first k terms of the series $S_k=\sum_{n=1}^{k}a_n$ (called the kth partial sum of the series).
- $S_\infty=\sum_{n=1}^{\infty}a_n=\lim_{k\to\infty}S_k$. If this limit exists then the infinite series is said to converge, otherwise the infinite series diverges.
- Informally, this means that if the series converges then as the value of k gets larger and larger, the value of S_k will approach (get closer and closer to and by taking k large enough can be made as close as we like to) a particular finite value, the limit. This limit is defined to be the sum of the infinite series. If S_k does not approach a finite value as k gets larger and larger, the limit does not exist and so the series diverges.

Convergence of the zeta function

$\frac{1}{n^s}\geq1$ for n any positive integer and $s\leq0$. Thus when $s\leq0$

$\sum_{n=1}^{\infty}\frac{1}{n^s}\geq\sum_{n=1}^{\infty}1=\infty$ (we write "$=\infty$" to mean that the series diverges to infinity). This series diverges since we can make the kth partial sum of the series larger than any given number by taking k large enough – this is what we mean when we say the series diverges to infinity. The zeta function must also therefore diverge to infinity for $s\leq0$.

For $s>0$ each term of the zeta function is less than the previous term (we say that the terms are decreasing).
Using this for $s>0$

$$1+\left(\frac{1}{2^s}+\frac{1}{3^s}\right)+\left(\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\frac{1}{7^s}\right)+\cdots\leq1+\left(\frac{1}{2^s}+\frac{1}{2^s}\right)+\left(\frac{1}{4^s}+\frac{1}{4^s}+\frac{1}{4^s}+\frac{1}{4^s}\right)+\cdots$$

We group the terms as above so that there are $1,2,4,8,16,\ldots$ terms in each successive group. Each group with 2^n terms in the series on the left-hand side is less than or equal to the group with 2^n terms in the series on the right-hand side. Grouping the terms in this way also allows us to see that the series on the right-hand-side is a geometric series.

Using similar logic we also have

$$\frac{1}{2^s}+\left(\frac{1}{4^s}+\frac{1}{4^s}\right)+\left(\frac{1}{8^s}+\frac{1}{8^s}+\frac{1}{8^s}+\frac{1}{8^s}\right)+\cdots\leq1+\left(\frac{1}{2^s}+\frac{1}{3^s}\right)+\left(\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\frac{1}{7^s}\right)+\cdots$$

Writing this in mathematical notation we obtain

$$\sum_{k=0}^{\infty}2^k\frac{1}{(2^{k+1})^s}\leq\sum_{n=1}^{\infty}\frac{1}{n^s}\leq\sum_{k=0}^{\infty}2^k\frac{1}{(2^k)^s}$$

The series on the far left-hand side and the far right-hand side are both geometric series with the zeta function in the centre.

Let $S_\infty=\sum_{k=0}^{\infty}2^k\frac{1}{(2^k)^s}=\sum_{k=0}^{\infty}\left(\frac{2}{2^s}\right)^k$. s will be a fixed value so the common ratio is $\frac{2}{2^s}$. For $s>1$, $\frac{2}{2^s}<1$ so we can apply the geometric series formula you will learn, obtaining $S_\infty=\frac{1}{1-\left(\frac{2}{2^s}\right)}$ and so the series converges in this case.

For $0<s\leq1$, $\frac{2}{2^s}\geq1\therefore\sum_{k=0}^{\infty}\left(\frac{2}{2^s}\right)^k\geq\sum_{k=0}^{\infty}1=\infty$ and so the series diverges to infinity in this case.

The same logic shows that $\sum_{k=0}^{\infty}2^k\frac{1}{(2^{k+1})^s}$ converges for $s>1$ and diverges to infinity for $0<s\leq1$.

For $0<s\leq1$ $\zeta(s)\geq\sum_{k=0}^{\infty}2^k\frac{1}{(2^{k+1})^s}=\infty$ and so the zeta function must also diverge to infinity for $0<s\leq1$.

For $s>1$ $\zeta(s)\leq S_\infty$ which converges. It can be shown that this, along with the fact that each term of the zeta function is positive, means that the zeta function must also converge for $s>1$.

In particular, this shows that $\zeta(1)=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n}$ diverges to infinity and also that $\zeta(s)$ is equal to a finite number when s is a positive integer (or any real number) greater than 1. Thus we define the zeta function for $s>1$, as we did earlier, so that the infinite series converges.

The Basel problem

We therefore know $\zeta(2)$ is convergent and will thus be equal to some finite number. The Basel problem, named after the Swiss city whose university the mathematicians and brothers Jakob and Johann Bernoulli attended, asked for the exact value of $\zeta(2)$. Jakob Bernoulli had asked for the solution of the problem in his book which showed his proof of the divergence of $\zeta(1)$.

The problem became well known in France and England around 1650 when it was published in the book "Novae Quadratura Arithmeticae" written by Pietro Mengoli and went on to achieve great mystique. The problem resisted attempts at solution by some of the greatest mathematicians of the time including Johann Bernoulli, John Wallis and Gottfried Leibniz, co-inventor of the calculus.

The problem was solved by Leonhard Euler in 1735. It was his first significant mathematical achievement and helped to create his great reputation.

Leonhard Euler

Leonhard Euler was born in Basel, Switzerland but spent most of his life at the Berlin Academy and the Imperial Russian Academy of Sciences in St.Petersburg. Euler is regarded as one of the greatest and most prolific mathematicians of all time.

He made significant contributions to many branches of mathematics, and, despite being totally blind for the last twelve years of his life, remained mathematically active until his death. Among other things, he is well known for one of the most famous formulae in mathematics, $e^{i\pi}=-1$ where i is the square root of -1. We shall see some of his other great accomplishments involving the zeta function.

Euler's solution of the Basel problem

Background

- a polynomial equation has the form

$a_nx^n+\cdots+a_1x+a_0=0$ where $a_n,\ldots,a_0$ are real numbers.

- n is called the degree of the polynomial. The a_0 term is named the constant term and the a_1x term is called the linear term
- A second degree polynomial equation can be written in the form $(x-r_1)(x-r_2)=0$ where r_1 and r_2 are the roots of the polynomial equation. If both roots are non-zero we can expand and divide by r_1r_2 to obtain

$$\frac{1}{r_1r_2}x^2-\left(\frac{1}{r_1}+\frac{1}{r_2}\right)x+1=0$$

In general, for a polynomial of finite degree, if the constant term is 1 then the linear term has coefficient equal to minus the sum of the reciprocals of the roots. Therefore $\frac{a_1}{a_0}=-\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\cdots+\frac{1}{r_n}\right)$ if $a_0\neq0$ and $r_1,\cdots,r_n$ are the roots of the polynomial equation

- A power series has the form $g(x)=\sum_{n=0}^{\infty}c_nx^n$ where the coefficients c_n are real numbers
- A function $f(x)$ which can be differentiated infinitely many times in an interval containing 0, such as $\sin x$, can be expressed as a power series of the form

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots$$

called the Maclaurin series of the function. It is likely that you will derive some Maclaurin series if you do A level further mathematics or a university calculus course.

Euler's solution [1]

Euler begins his solution with the Maclaurin series

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n+1}}{(2n+1)!}$$

where x is measured in radians. This series converges for all x.

Taking $x\neq0$ we can divide by x to obtain

$$\frac{\sin x}{x}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots$$

which Euler treats as an infinite polynomial.

Taking $f(x)=\frac{\sin x}{x}=0$ the equation has roots $\pm\pi,\pm2\pi,\pm3\pi,\ldots$ but not $x=0$ since, as you will learn in calculus, $\lim_{x\to0}\frac{\sin x}{x}=1$.

Making the substitution $u=x^2$ we obtain

$$1-\frac{u}{3!}+\frac{u^2}{5!}-\frac{u^3}{7!}+\cdots=0$$

which has roots $\pi^2,4\pi^2,9\pi^2,16\pi^2,\ldots$

Euler then applied the reasoning above related to the sum of reciprocals of roots to this infinite polynomial obtaining

$$-\frac{1}{3!}=-\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\frac{1}{16\pi^2}+\cdots\right)$$

Multiplying through by $-\pi^2$ we obtain

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{6}\text{ therefore }\zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

This is one of the most celebrated results in mathematics. Furthermore, the methods Euler used allowed him to calculate $\zeta(s)$ for further even values of s and in 1740 he proved that for any positive integer n, $\zeta(2n)=a_{2n}\pi^{2n}$ where a_{2n} are rational numbers.

However, Euler's proof of the Basel problem would not be considered mathematically rigorous today in the sense that his reasoning about the sum of reciprocals of roots with an infinite polynomial (we know it would be correct for a polynomial of finite degree) would need to be justified for the entire proof to be accepted in modern mathematics.

The result obtained by Euler is correct nevertheless, and can be proved in several different ways. Euler proved it again in 1741 using calculus and Maclaurin series. The exact values of $\zeta(s)$ when s is an odd positive integer remains a mystery though.

Physical problem revisited

We have now established the properties of the zeta function needed to solve the physical problem we raised earlier.

Recall that we found the force exerted on the yellow particle due to the infinitely many green particles was $G\zeta(2)$ Newtons where G is the universal gravitational constant.

From our analysis of the convergence of the zeta function we now know that as we sum the attractive force each successive green particle exerts on the yellow particle, this sum will converge to some finite value since $\zeta(2)$ is convergent.

Furthermore, from Euler's solution of the Basel problem we know $\zeta(2)=\frac{\pi^2}{6}$ and so the force exerted on the yellow particle due to the infinitely many green particles is

$G\frac{\pi^2}{6}$ Newtons.

Since the numerical value of G is 6.67×10^{-11} the numerical value of this force to 3 significant figures is

1.10×10^{-10} Newtons.

Thus the magnitude of the force exerted on the yellow particle due to the green particles is extremely small and so it can be regarded as insignificant – as we might expect.

Further reading

- "Euler's Solution of the Basel Problem – The Longer Story" by Ed Santifer gives three of Euler's proofs of the Basel problem along with historical background and can be accessed at http://www.southernct.edu/~sandifer/Ed/History/Preprints/Talks/NYU%20Basel%20Problem%20Paper.PDF
- "Prime obsession" by John Derbyshire. This book explores the zeta function in the wider context of the Riemann hypothesis. Historical and mathematical aspects of the Riemann hypothesis are both covered in the book.

[1] Source of proof: "Analysis" by P E Kopp

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